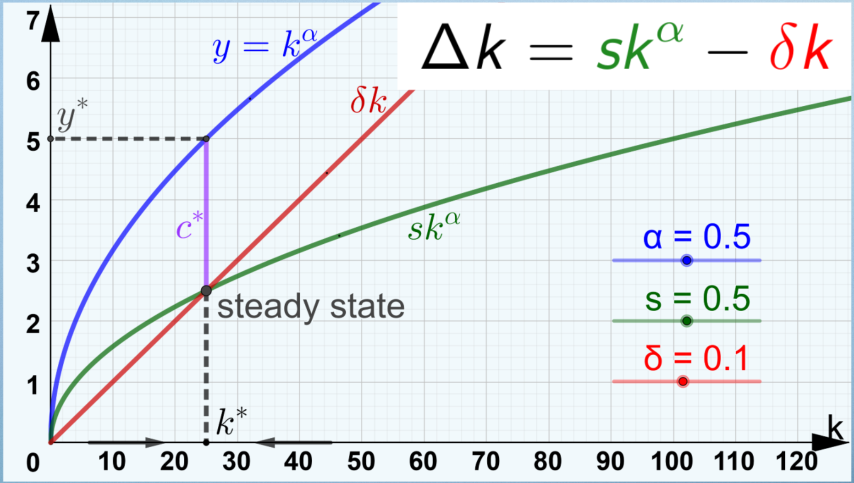
SOLOW MODEL



$\Delta k = sk^{\alpha} - \delta k$
$y = k^{\alpha}$
δk
sk^{α}
y^*
c^*
k^*
steady state
α = 0.5
s = 0.5
δ = 0.1
k
0
1
2
3
4
5
6
7
10
20
30
40
50
60
70
80
90
100
110
120

Production function

$$\text{Inputs} \begin{cases} \text{Capital } K \\ \text{Labor } N \end{cases} \Rightarrow \quad \text{Output } Y$$

Cobb-Douglas production function

$$Y = F(K, N) = K^{\alpha} N^{1-\alpha} \qquad (\text{with } 0 < \alpha < 1)$$

Constant returns to scale

Double **all** inputs $\Rightarrow$ double output

Positive but diminishing returns (marginal products) for both inputs

If we hold **one** input (e.g., labor N) constant and increase the **other input** (e.g., capital)...

- our output increases (**positive marginal product**)
- the increase of the output diminishes with each additional unit of this input (**diminishing marginal product**)

How does income per capita evolve over time?

Not the total income Y, but the income per capita $y = Y/N$ explains standard of living!

Definition

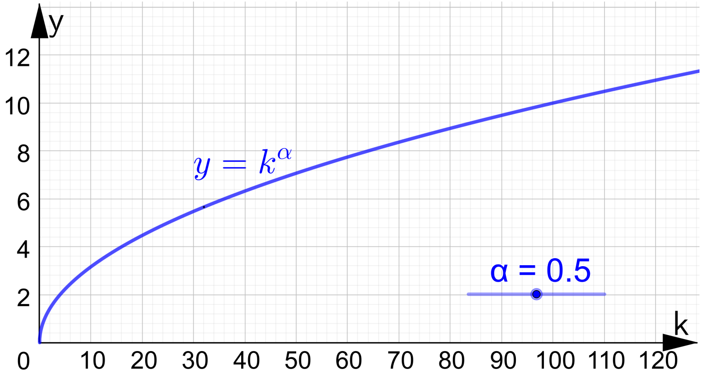
Aggregate (absolute) variables use **capital letters**, while **per-capita variables** use **lower case** letters.

$$
\begin{aligned}
Y &= K^{\alpha} N^{1-\alpha} \\
\frac{Y}{N} &= \frac{K^{\alpha} N^{1-\alpha}}{N} \\
y &= K^{\alpha} N^{-\alpha} \\
y &= \frac{K^{\alpha}}{N^{\alpha}} \\
y &= k^{\alpha}
\end{aligned}
$$

$\Rightarrow$ **income per capita** y only depends on **capital per capita** k!

Production function in intensive form

- $y = f(k) = k^\alpha$ is the production function in **intensive form**
- $f(k) = k^\alpha$ is **increasing** in k, $f'(k) > 0$
- $f(k) = k^\alpha$ is **concave**, $f''(k) < 0$ (becomes „flatter“)



Capital accumulation

- **Investments** I_t add to the total capital stock
- Closed economy (and without government, $G = 0,\ T = 0$)

$$\text{investments } I_t = \text{savings } S_t$$

- Households **save the fixed (exogenous) share** s of their income $Y_t \Rightarrow S_t = sY_t$
- Share δ capital K_t gets **destroyed** by $\Rightarrow$ **depreciation** δK_t
- Capital stock in the next period, K_{t+1}, is

$$\begin{aligned} K_{t+1} &= K_t + sY_t - \delta K_t \\ K_{t+1} - K_t &= sY_t - \delta K_t \\ \Delta K &= sY - \delta K \end{aligned}$$

$$\textbf{net investment} = \textbf{gross investment} - \textbf{depreciation}$$

How does capital per capita k evolve over time?

Aggregate capital stock evolves according to

$$\Delta K = sY - \delta K$$

We want:
Income **per capita** $y = f(k) = k^{\alpha}$!

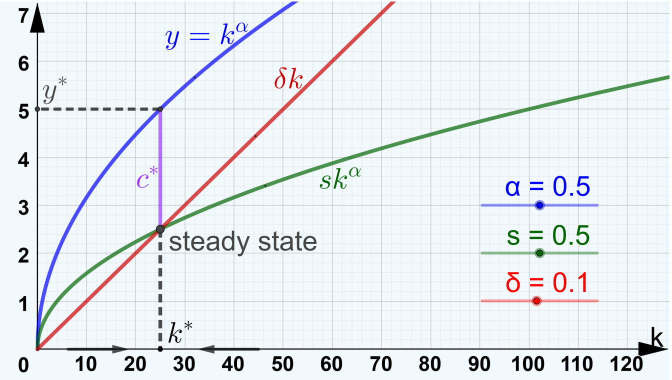
$$\begin{aligned} \frac{\Delta K}{N} &= \frac{sY - \delta K}{N} \\ \Delta k &= sy - \delta k \end{aligned}$$

Dynamic evolution of capital per worker in the Solow model

$$\Delta k = sk^{\alpha} - \delta k$$

Dynamics and steady state

$$\Delta k = sk^{\alpha} - \delta k$$



Steady state

In the **steady state** (long-run equilibrium) k^*, **capital per capita is constant**.

In the **steady state**, **investment** per capita must be equal to **depreciation** per capita:

$$sk^{\alpha} = \delta k$$

This allows us to calculate the **capital per capita** in the steady state, k^*:

$$\begin{aligned} s &= \delta k^{1-\alpha} \\ k^{1-\alpha} &= \frac{s}{\delta} \end{aligned}$$

Capital per capita in the steady state, k^*

$$k^* = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$$

Income per capita in the steady state, y^*

$$y^* = k^{*\alpha} = \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Consumption per capita in the steady state, c^*

$$c^* = (1-s)y^* = (1-s)\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Increase of savings rate from $s = 0.5$ to $s = 0.6$

$$\Delta k = sk^{\alpha} - \delta k$$